

c–Map, very Special Quaternionic Geometry and Dual Kähler Spaces.

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ABSTRACT

We show that for all very special quaternionic manifolds a different $N = 1$ reduction exists, defining a Kähler Geometry which is “dual” to the original very special Kähler geometry with metric $G_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \ln V$ ($V = \frac{1}{6} d_{abc} \lambda^a \lambda^b \lambda^c$). The dual metric $g^{ab} = V^{-2} (G^{-1})^{ab}$ is Kähler and it also defines a flat potential as the original metric. Such geometries and some of their extensions find applications in Type IIB compactifications on Calabi–Yau orientifolds.

1 Isometries of dual quaternionic manifolds

One of the basic constructions in dealing with the low energy effective Lagrangians of Type IIA and Type IIB superstrings is the so called c -map [1], which associates to any Special Kähler manifold of complex dimension n a “dual” quaternionic manifold of quaternionic dimension $n_H = n + 1$.

In particular it was shown [2] that “dual” quaternionic manifolds always have at least $2n + 4$ isometries: one scale isometry ϵ_0 and $2n + 3$ shift isometries $\beta_I, \alpha^I, \epsilon_+$ ($I = 0, \dots, n$), whose generators close a Heisenberg algebra [3]:

$$[\beta^I, \epsilon^+] = [\alpha_I, \epsilon^+] = 0; \quad [\beta^I, \alpha_J] = \delta_J^I \epsilon^+; \quad [\epsilon^0, \alpha_I] = \frac{1}{2} \alpha_I; \quad [\epsilon^0, \beta^I] = \frac{1}{2} \beta^I; \quad [\epsilon^0, \epsilon^+] = \epsilon^+ \quad (1.1)$$

The corresponding generators can be written according to their ϵ^0 weight as [4, 5, 6, 7]:

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} + \mathcal{V}_1. \quad (1.2)$$

However it was shown in [6, 7] that when the Special Kähler manifold has some isometries, then some “hidden symmetries” are generated in the c -map spaces which are classified by $\mathcal{V}_{-1}, \mathcal{V}_{-\frac{1}{2}}$, with

$$\dim(\mathcal{V}_{-1}) \leq 1; \quad \dim(\mathcal{V}_{-\frac{1}{2}}) \leq 2n + 2. \quad (1.3)$$

In particular, for a generic very special geometry, with a cubic polynomial prepotential

$$F(z) = \frac{1}{48} d_{abc} z^a z^b z^c \quad (1.4)$$

with generic d_{abc} , with no additional isometries, it was shown that:

$$\dim(\mathcal{V}_{-1}) = 0; \quad \dim(\mathcal{V}_{-\frac{1}{2}}) = 1; \quad \dim(\mathcal{V}_0) = n + 2. \quad (1.5)$$

Since the isometries of a generic very Special Geometry of dimension n are $n + 1$, the dual manifold has then $3n + 6$ isometries, where the $n + 2$ additional isometries lie, $n + 1$ in \mathcal{V}_0 , denoted by ω_I , ($I = 0, \dots, n$), and one $\hat{\beta}_0$ in $\mathcal{V}_{-\frac{1}{2}}$. For symmetric spaces the upper bound in equation (1.3) is saturated so that $\dim G_Q = \dim G_{SK} + 4n + 7$ where G_{SK} and G_Q are the isometry groups of the Special Kähler and Quaternionic spaces respectively.

2 The very Special σ -model Lagrangian and its $N = 1$ reduction

The quaternionic “dual” σ -model for a generic Special Geometry was derived in [2] by dimensional reduction of a $N = 2$ Special Geometry to three dimensions. By adapting the

conventions of [2] to those of [6] and [8] we call the special coordinates z^a as $z^a = x^a + iy^a$ and define:

$$\begin{aligned} V &= \frac{1}{6}(\kappa yyy) \equiv \frac{1}{6}\kappa & (\kappa yyy) &= d_{abc}y^a y^b y^c \\ \kappa_a &= d_{abc}y^b y^c; & \kappa_{ab} &= d_{abc}y^c \end{aligned} \quad (2.1)$$

The $2n + 4$ additional coordinates are denoted by $\zeta^I \equiv (\zeta^0, \zeta^a)$, $\tilde{\zeta}_I \equiv (\tilde{\zeta}_0, \tilde{\zeta}_a)$, $D, \tilde{\Phi}$.

The α^I, β_I isometries act as shifts on the $2n + 2$ coordinates $\zeta^I, \tilde{\zeta}_I$:

$$\delta\zeta^I = \alpha^I; \quad \delta\tilde{\zeta}_I = \beta_I \quad (2.2)$$

while the ω^a shift isometries of the special geometry, $\delta x^a = \omega^a$, act as duality rotations on the $\zeta^I, \tilde{\zeta}_I$ symplectic vector:

$$\delta \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & -A^T \end{pmatrix} \begin{pmatrix} \zeta \\ \tilde{\zeta} \end{pmatrix} \quad (2.3)$$

with

$$A = \begin{pmatrix} 0 & 0 \\ \omega^a & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 3d_{abc}\omega^c \end{pmatrix}. \quad (2.4)$$

On the other hand the $\hat{\beta}_0$ isometry rotates ζ^a into x^a so that the $x^a, \tilde{\zeta}_a$ variables are related by quaternionic isometries. It is immediate to see that the full σ -model Lagrangian [2, 6, 8] is invariant under the following parity operation Ω :

$$y^a \rightarrow y^a; \quad \tilde{\zeta}_a \rightarrow \tilde{\zeta}_a; \quad \zeta^0 \rightarrow \zeta^0; \quad D \rightarrow D; \quad (2.5)$$

$$x^a \rightarrow -x^a; \quad \zeta^a \rightarrow -\zeta^a; \quad \tilde{\zeta}_0 \rightarrow -\tilde{\zeta}_0; \quad \tilde{\Phi} \rightarrow -\tilde{\Phi} \quad (2.6)$$

so that, restricting to the plus-parity sector is a consistent truncation, giving rise to the following Lagrangian for $2n + 2$ (real) variables:

$$\begin{aligned} (\sqrt{-g})^{-1}\mathcal{L} &= -(\partial_\mu D)^2 - \frac{1}{4}G_{ab}\partial_\mu y^a \partial^\mu y^b - \frac{1}{8}e^{2D}V(\partial_\mu \zeta^0)^2 \\ &\quad - 2e^{2D}V^{-1}(G^{-1})^{ab}\partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b \end{aligned} \quad (2.7)$$

where $G_{ab} = -\partial_a \partial_b \log V$. By a change of variables we can decouple the (D, ζ^0) fields from the rest as follows: define two new variables (Φ, λ^a) :

$$V(y)e^{2D} = e^{2\Phi}; \quad y^a = \lambda^a e^{\frac{\Phi}{2}} \quad (2.8)$$

Thus it follows that $V(\lambda)e^{2D} = e^{\frac{\Phi}{2}}$ and the Lagrangian becomes:

$$\begin{aligned} (\sqrt{-g})^{-1}\mathcal{L} &= -\frac{1}{4}(\partial_\mu \Phi)^2 - \frac{1}{8}e^{2\Phi}(\partial_\mu \zeta^0)^2 - \frac{1}{4}G_{ab}\partial_\mu \lambda^a \partial^\mu \lambda^b \\ &\quad - \frac{1}{4}(\partial_\mu \log V(\lambda))^2 - 2V(\lambda)^{-2}(G^{-1})^{ab}\partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b \end{aligned} \quad (2.9)$$

the (Φ, ζ^0) part defines a $SU(1, 1)/U(1)$ σ -model.

The coefficient of the two terms in the $\partial_\mu \lambda^a \partial^\mu \lambda^b$ part combine into $-\frac{3}{2} \left(\frac{\kappa_{ab}}{\kappa} - 3 \frac{\kappa_a \kappa_b}{\kappa^2} \right)$. We now define a new variable $t_a = \frac{1}{2} \kappa_{ab} \lambda^b$ such that $d\lambda^b = (\kappa^{-1})^{ba} t_a$ we obtain that

$$g^{ab} = -6 \left(\frac{\kappa_{cd}}{\kappa} - 3 \frac{\kappa_c \kappa_d}{\kappa^2} \right) (\kappa^{-1})^{ac} (\kappa^{-1})^{bd} = -\frac{6}{\kappa^2} [(\kappa^{-1})^{ab} \kappa - 3 \lambda^a \lambda^b] = \frac{36}{\kappa^2} (G^{-1})^{ab} \quad (2.10)$$

Therefore in the $(t_a, \tilde{\zeta}_a)$ variables we finally get

$$\begin{aligned} (\sqrt{-g})^{-1} \mathcal{L} = & -\frac{1}{4} (\partial_\mu \Phi)^2 - \frac{1}{8} e^{2\Phi} (\partial_\mu \zeta^0)^2 - \frac{1}{4} g^{ab} \partial_\mu t_a \partial^\mu t_b \\ & - 2g^{ab} \partial_\mu \tilde{\zeta}_a \partial^\mu \tilde{\zeta}_b \end{aligned} \quad (2.11)$$

Therefore by defining the complex variables

$$\eta_a = t_a + 2\sqrt{2}i\tilde{\zeta}_a \quad (2.12)$$

we get for the $2n$ -dimensional σ -model:

$$-\frac{1}{4} g(\Re \eta)^{ab} (\partial_\mu \Re \eta_a \partial^\mu \Re \eta_b + \partial_\mu \Im \eta_a \partial^\mu \Im \eta_b) = -\frac{1}{4} g^{ab} \partial_\mu \eta_a \partial^\mu \bar{\eta}_b \quad (2.13)$$

The previous Lagrangian is Kähler provided

$$g(t)^{ab} = \frac{\partial^2 \hat{K}}{\partial t_a \partial t_b}. \quad (2.14)$$

This condition is achieved by setting $\hat{K} = -2 \log V(\lambda)$. Indeed

$$\begin{aligned} \frac{\partial}{\partial t_a} \log V &= (\kappa^{-1})^{ac} \frac{\partial}{\partial \lambda^c} \log V = 3 \frac{\lambda^a}{\kappa} \\ \frac{\partial^2}{\partial t_a \partial t_b} \log V &= 3 \left[\frac{(\kappa^{-1})^{ab}}{\kappa} - 3 \frac{\lambda^a \lambda^b}{\kappa^2} \right] = -\frac{1}{2} \times \frac{36}{\kappa^2} (G^{-1})^{ab} \end{aligned} \quad (2.15)$$

3 Isometries of the $N = 1$ reduction

The σ -model isometries of the c-map, using the notations of [7] are parametrized by

$$\epsilon^+, \epsilon^0, \alpha^I, \beta_I, \omega^a, \omega^0, \hat{\beta}_0. \quad (3.1)$$

The $N = 1$ reduction projects out $\epsilon^+, \alpha^a, \beta_0, \omega^a$, so the remaining isometries are $n + 4$, namely:

$$\beta_a, \omega^0, \epsilon^0, \alpha^0, \hat{\beta}_0. \quad (3.2)$$

Three of the latter generate a $SL(2, \mathbb{R})$ symmetry (otherwise absent in generic dual quaternionic manifolds), the others generate a shift symmetry in $\Im \eta_a$ and a scale symmetry in the

η_a variables. The dual manifold has the same isometries of the original Special Kähler. Since the $\tilde{\zeta}_a$ variables are related to the x^a variables by quaternionic isometries, the two manifolds need in fact not be distinct. Even though the $\tilde{\zeta}_a$ variables are related to the x^a variables by quaternionic isometries, the two manifolds are in general distinct. However, in the particular case of homogeneous-symmetric spaces [9], it turns out that the dual manifold coincide with the original one. The proof of this statement will be given elsewhere.

4 Connection with Calabi Yau orientifolds

The c-map was originally studied in relation to the Type II A \rightarrow Type II B mirror map in Calabi-Yau compactifications. In Calabi Yau orientifolds of Type II B strings with D-branes present, the bulk Lagrangian is obtained combining a world-sheet parity with a manifold parity which, for generic spaces [10], is precisely doing the truncation we have encountered in this note.

For certain Calabi Yau manifolds more generic orientifoldings are possible where the set of special coordinates z^A is separated in two parts with opposite parity, z_{\pm}^A ($n_+ + n_- = n$) such that [11]

$$\begin{aligned} y_{\pm} &\rightarrow \pm y_{\pm} \\ x_{\pm} &\rightarrow \mp x_{\pm} \end{aligned} \quad (4.1)$$

and then consequently

$$\begin{aligned} \zeta_{\pm} &\rightarrow \mp \zeta_{\pm} ; \quad \zeta^0 \rightarrow \zeta_0 \\ \tilde{\zeta}_{\pm} &\rightarrow \pm \tilde{\zeta}_{\pm} ; \quad \tilde{\zeta}_0 \rightarrow -\tilde{\zeta}_0 \end{aligned} \quad (4.2)$$

However in this case one must demand

$$d_{++-} = d_{---} = 0 \quad (4.3)$$

in order for the $N = 1$ reduction to be consistent [12].

In this case the σ -model Lagrangian acquires more terms and can be symbolically written as:

$$\begin{aligned} (\sqrt{-g})^{-1} \mathcal{L} &= -(\partial D)^2 - \frac{1}{4} G_{++} (\partial y_+)^2 - \frac{1}{4} G_{--} (\partial x_-)^2 - \frac{1}{8} e^{2D} V (\partial \zeta^0)^2 - \\ &\quad \frac{1}{8} e^{2D} V G_{--} (x_- \partial \zeta^0 - \partial \zeta_-)^2 - \\ &\quad 2e^{2D} V^{-1} (G^{-1})^{++} (\partial \tilde{\zeta}_+ + \frac{1}{8} d_{+--} x_- \partial \zeta^0 - \frac{1}{4} d_{+--} x_- \partial \zeta_-)^2 \end{aligned} \quad (4.4)$$

where for the sake of simplicity space–time indices have been suppressed from partial derivatives and contraction over them is understood. In (4.4) G_{++} is as before since $d_{+++} \neq 0$, $G_{+-} = 0$ and $G_{--} = -6(d_{--+}y_+)/ (d_{+++}y_+y_+y_+)$.

The total set of coordinates are: y_+ , x_- , ζ_- , $\tilde{\zeta}_+$ and (Φ, ζ^0) . Since in this case some of the y coordinates, namely y_- , have been replaced by x_- , the new variables define a Kähler manifold of complex dimension $n + 1$ certainly distinct from the original one.

There is an $N = 4$ analogue of this dual $N = 1$ geometries if we consider different embeddings of $N = 4$ supergravity into $N = 8$. This corresponds to Type II B on T^6/\mathbb{Z}_2 orientifold with $D3$ or $D9$ branes (Type I string) or Heterotic string on T^6 . In all these cases the bulk sector corresponds to $[\text{SO}(6, 6)/\text{SO}(6) \times \text{SO}(6)] \times [\text{SU}(1, 1)/\text{U}(1)]$ σ –model but the 15 axions in $\text{SO}(6, 6)/\text{SO}(6) \times \text{SO}(6)$ are coming from C_4 , C_2 , B_2 [13, 14, 15, 16, 17, 18, 19].

Also cases in which a further splitting appears are realized if the orientifold projection acts [18] differently on $T^{p-3} \times T^{9-p}$ ($p = 3, 5, 7, 9$). This is the analogue of the y_{\pm} , x_{\pm} splitting [11]. In all these cases the dual manifolds coincide, as predicted by $N = 4$ supergravity.

5 Properties of the dual Special Kähler spaces and no–scale structure.

The dual Kähler space, obtained by a $N = 1$ truncation of the (c–map) very special quaternionic space has a metric that satisfies a “duality” relation with the original very special Kähler space:

$$g_D^{ab} = \frac{1}{V^2} (G^{-1})^{ab} \quad (5.1)$$

Moreover it can be shown that its affine connection is simply related to the affine connection of original Kähler space:

$$\Gamma_d^{Dbc} = \frac{1}{V} (G^{-1})^{ca} \Gamma_{ad}^b. \quad (5.2)$$

Actually in the one–dimensional case the two connections coincide.

These dual spaces are also no–scale [20, 21, 22]. Indeed it is sufficient to prove that

$$\frac{\partial \hat{K}}{\partial \Re \eta_a} (g^{-1})_{ab} \frac{\partial \hat{K}}{\partial \Re \eta_b} = 3. \quad (5.3)$$

But this is indeed the case since

$$\lambda^a G_{ab} \lambda^b = 3. \quad (5.4)$$

From a Type II B perspective, this was anticipated in [23]

6 Concluding remarks.

In this note we have shown that for an arbitrary very special geometry, through the c-map, it is possible to construct a “dual” Kähler geometry which has a dual metric, it is Kähler and it provides a dual no-scale potential. Recently such constructions have found applications in Calabi Yau orientifolds [24, 11] but the procedure considered here is intrinsic to the four dimensional context.

We have not shown that the final Lagrangian is supersymmetric but, using the reduction techniques of [12], it can be shown that this is indeed the case. It is reassuring that the $SL(2, \mathbb{R})$ symmetry, related to the Type II B interpretation, comes out in a pure four dimensional context, thanks to the results of [6, 7]

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